Infinitesimal Generators of Markovian Stochastic Fluid Flow Models

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Outline

CTMC and its Generator Matrix $\mathbf{T}$

- Let $\{\varphi(t), t \geq 0\}$ be an irreducible CTMC with a (finite) set $\mathcal{S}$ of all possible *phases* $\varphi(t)$
- Let $P(t)_{ij} = P(\varphi(t) = j | \varphi(0) = i)$ and $\mathbf{P}(t) = [P(t)_{ij}]$
- Note that $\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$ and $\lim_{h \to 0^+} \mathbf{P}(h) = \mathbf{I}$

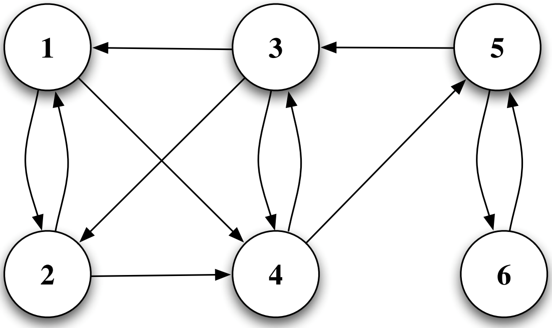
Fact

$$\mathbf{P}(t) = e^{\mathbf{T}t}$$

with generator

$$\mathbf{T} = \lim_{h \to 0^+} \frac{\mathbf{P}(h) - \mathbf{I}}{h}$$

Example - Hydro-Power Generator



1 on-design, 2 off-design, 3 start, 4 stop, 5 idle, 6 maintenance

Stationary Distribution Vector π

- Assume that the CTMC is positive-recurrent
- Let $\pi_j = \lim_{t\to\infty} P(t)_{ij}$

Fact

$\boldsymbol{\pi} = [\pi_i]_{i\in\mathcal{S}}$ is the solution of

$$\begin{cases} \pi\mathbf{T} & = & \mathbf{0} \\ \pi\mathbf{1} & = & 1 \end{cases}$$

*where **0** is a row vector of zeros, **1** is a column vector of ones*

Example - Hydro-Power Generator

Given

- CTMC with $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and generator $\mathbf{T}$
- Revenue rate c_i for all $i \in \mathcal{S}$

we can derive

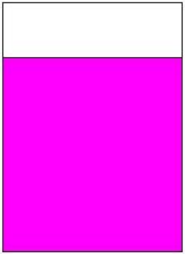
$$\text{long-run mean revenue} = \sum_i \pi_i c_i$$

But we would like to do better than this!

Definition of a 1-D FM

Let $\{(\varphi(t), Y(t)), t \geq 0\}$ be a process such that:

- $\{\varphi(t), t \geq 0\}$ is an irreducible CTMC with a (finite) set of phases $\mathcal{S}$ and generator $\mathbf{T}$
- $\{\varphi(t), t \geq 0\}$ is the driving process
- *Level* $Y(t)$ records some performance measure
- When $\varphi(t) = i$, the rate at which $Y(t)$ is changing is r_i



Buffer Y

$$\frac{dY(t)}{dt} = r_i \quad \text{when} \quad \varphi(t) = i \ \text{ and } \ Y(t) > 0$$

Example - Hydro-Power Generator

To model the deterioration process, let

- $Y(t) \in [0, 1]$ be the deterioration level
- 0 - brand new, 1 - needs replacement
- r_i be deterioration rates, $i \in \{1, 2, 3, 4, 5, 6\}$

In-Out Fluid Idea

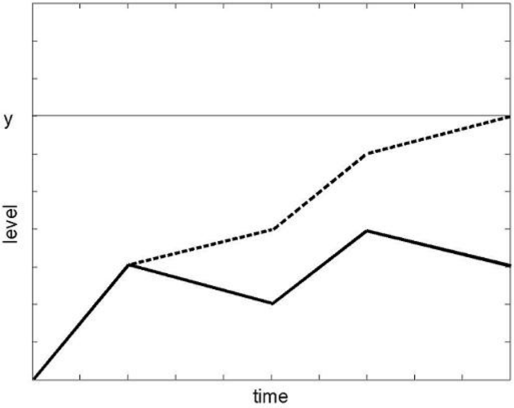


Figure: Start in $(i, 0)$, end in (j, y) at time $\hat{\theta}(y)$

Corresponding Laplace-Stieltjes Transform (LST)

Let

- $|Y(t)| = \int_{u=0}^{t} |r_{\varphi(u)}| du$
- $\hat{\theta}(y) = \inf\{t \geq 0 : |Y(t)| = y\}$

Definition

Let $\hat{\Delta}^y(s) = [\hat{\Delta}^y(s)_{ij}]$ be such that for all $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$

$$\hat{\Delta}^y(s)_{ij} = E(e^{-s\hat{\theta}(y)} : \varphi(\hat{\theta}(y)) = j | \varphi(0) = i, Y(t) = 0)$$

Some Notation

- $\mathcal{S}_1 = \{i \in \mathcal{S} : r_i > 0\}$
- $\mathcal{S}_2 = \{i \in \mathcal{S} : r_i < 0\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : r_i = 0\}$
- $\mathbf{R}_1 = diag(r_i)$ for all $i \in \mathcal{S}_1$
- $\mathbf{R}_2 = diag(|r_i|)$ for all $i \in \mathcal{S}_2$

- $\mathbf{T}_{11} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_1$
- $\mathbf{T}_{12} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_2$
- $\mathbf{T}_{10} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_0$
- etc

Matrix $\mathbf{Q}(s)$: assume $Re(s) \geq 0$

$$
\begin{aligned}
\mathbf{Q}_{11}(s) &= \mathbf{R}_1^{-1}[(\mathbf{T}_{11} - s\mathbf{I}) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}] \\
\mathbf{Q}_{22}(s) &= \mathbf{R}_2^{-1}[(\mathbf{T}_{22} - s\mathbf{I}) - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}] \\
\mathbf{Q}_{12}(s) &= \mathbf{R}_1^{-1}[\mathbf{T}_{12} - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}] \\
\mathbf{Q}_{21}(s) &= \mathbf{R}_2^{-1}[\mathbf{T}_{21} - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}]
\end{aligned}
$$

Definition

$$
\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{11}(s) & \mathbf{Q}_{12}(s) \\ \mathbf{Q}_{21}(s) & \mathbf{Q}_{22}(s) \end{bmatrix}
$$

$$
\mathbf{Q} = \mathbf{Q}(0) = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}
$$

$\mathbf{Q}(s)$ as the Generator of the 1-D FM

Theorem

$$\hat{\Delta}^{y}(s) = e^{\mathbf{Q}(s)y}$$

Proof.

- Note that $\hat{\Delta}^{y+u}(s) = \hat{\Delta}^{y}(s)\hat{\Delta}^{u}(s)$ and $\lim_{y\to 0^+} \hat{\Delta}^{y}(s) = \mathbf{I}$
- Evaluate $\hat{\Delta}^{h}(s)$ for small h
- Show that $\lim_{h\to 0^+} \frac{\hat{\Delta}^{h}(s)-\mathbf{I}}{h} = \mathbf{Q}(s)$

□

$\mathbf{Q}_{11}(s)$, $\mathbf{Q}_{22}(s)$ as Generators

Note the meaning of

- $e^{\mathbf{Q}_{11}(s)y}$
- $e^{\mathbf{Q}_{22}(s)y}$

as LSTs

Return to Level Zero in $Y(.)$

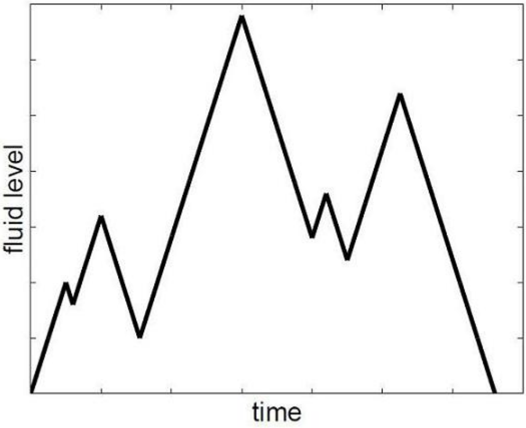


Figure: Start in $(i, 0)$, end in $(j, 0)$ at time $\theta(0)$

Matrix $\mathbf{\Psi}(s)$

Let $\theta(0) = \inf\{t \geq 0 : Y(t) = 0\}$

Definition

For s with $Re(s) \geq 0$*, i with* $r_i > 0$*, j with* $r_j < 0$*, let*

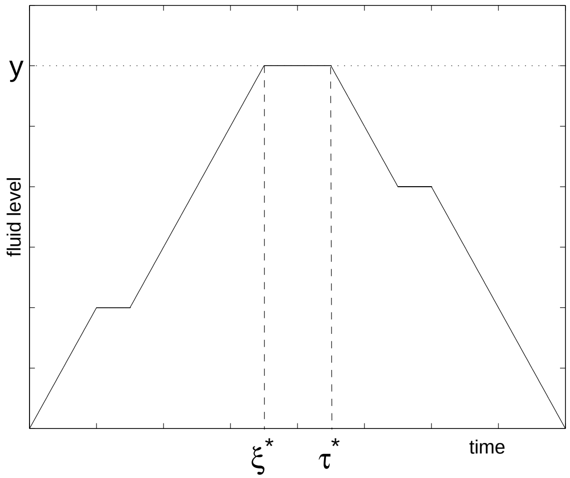
$$\Psi(s)_{ij} = E(\theta(0) < \infty, \theta(0) = i|\varphi(0) = i, Y(0) = 0)$$

Let

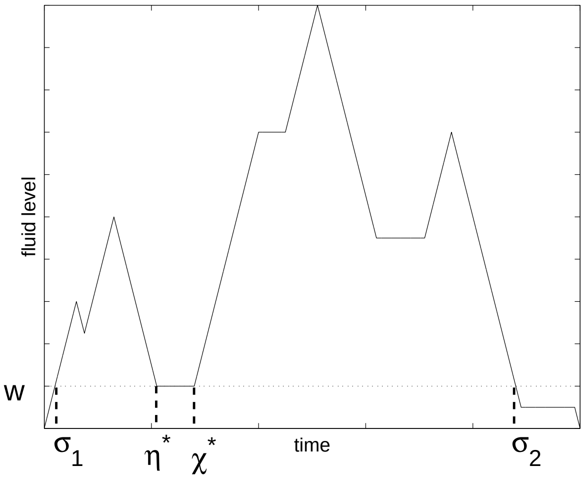
$$\mathbf{\Psi}(s) = [\Psi(s)_{ij}]$$

$$\mathbf{\Psi} = \mathbf{\Psi}(0) = [\Psi_{ij}]$$

Two types of returns: 1) No down-up periods



Two types of returns: 2) With down-up periods



Integral Equation for $\mathbf{\Psi}$

Fact

$$\mathbf{\Psi}(s) = \int_{y=0}^{\infty} e^{\mathbf{Q}_{11}(s)y} \left(\mathbf{Q}_{12}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{21}(s)\mathbf{\Psi}(s) \right) e^{\mathbf{Q}_{22}(s)y} dy$$

Riccati Equation for $\mathbf{\Psi}$

Fact

For $s \geq 0$, $\mathbf{\Psi}(s)$ is the minimum nonnegative solution of

$$\mathbf{Q}_{12}(s) + \mathbf{Q}_{11}(s)\mathbf{\Psi}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{22}(s) + \mathbf{\Psi}(s)\mathbf{Q}_{21}(s)\mathbf{\Psi}(s) = \mathbf{0}$$

Efficient Algorithm for $\mathbf{\Psi}(s)$

- Let $\mathbf{\Psi}(s,0) = \mathbf{0}$
- For $n \geq 1$, evaluate $\mathbf{\Psi}(s,n+1)$, by solving

$$\mathbf{A}\mathbf{\Psi}(s,n+1) + \mathbf{\Psi}(s,n+1)\mathbf{B} = \mathbf{C}$$

where

$$\begin{aligned}
\mathbf{A} &= \mathbf{Q}_{11}(s) + \mathbf{\Psi}(s,n)\mathbf{Q}_{21}(s) \\
\mathbf{B} &= \mathbf{Q}_{22}(s) + \mathbf{Q}_{21}(s)\mathbf{\Psi}(s,n) \\
\mathbf{C} &= -\mathbf{Q}_{12}(s) + \mathbf{\Psi}(s,n)\mathbf{Q}_{21}(s)\mathbf{\Psi}(s,n)
\end{aligned}$$

until a stopping criterion is met

(Transient) Results Derived Using $\mathbf{Q}(s)$ and $\mathbf{\Psi}(s)$

- Return to the original level
- Draining/Filling to some level
- Return to the original level while avoiding some taboo level
- Draining/Filling to some level while avoiding some taboo level
- Treatment of models with unbounded, bounded and multi-layer buffers

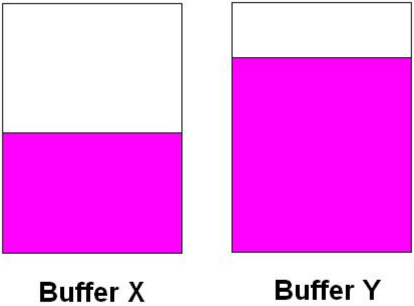
Definition of a 2-D FM

Let $\{(\varphi(t), X(t), Y(t)), t \geq 0\}$ be a process such that

- $\{(\varphi(t), X(t)), t \geq 0\}$ is a 1-D FM with the set of phases $\mathcal{S}$, generator $\mathbf{T}$ and rates c_i
- $\{(\varphi(t), Y(t)), t \geq 0\}$ is a 1-D FM with the set of phases $\mathcal{S}$, generator $\mathbf{T}$ and rates r_i

We study the case

- $X(t) \in (-\infty, +\infty)$
- $Y(t) \geq 0$



$$\frac{dX(t)}{dt} = c_i \quad \text{when} \quad \varphi(t) = i$$

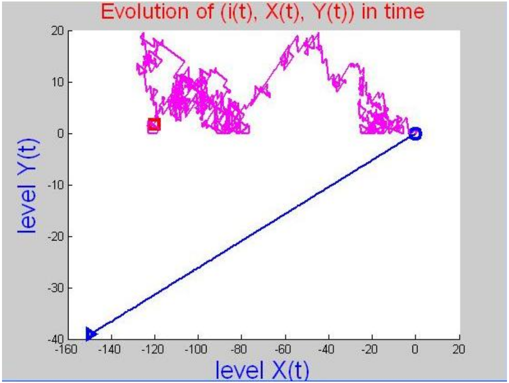
$$\frac{dY(t)}{dt} = r_i \quad \text{when} \quad \varphi(t) = i \ \text{ and } \ Y(t) > 0$$

Example - Hydro-Power Generator

To model the deterioration and revenue processes, let

- $\varphi(t) \in \{1, 2, 3, 4, 5, 6\}$ be the phase at time t
- $X(t) \in (-\infty, +\infty)$ be the total revenue level, with rates c_i
- $Y(t) \in [0, 1]$ be the deterioration level, with rates r_i

Sample Path Example - $X(t) \in (-\infty, +\infty)$, $Y(t) \geq 0$



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Shift Idea

- Let $Z(t) = X(t) - X(0)$ be the total **shift** in $X(.)$ at time t
- Evaluate the LST of shift $Z(.)$ for a path of interest in $Y(.)$
- Then, given initial state $(\varphi(0), X(0), Y(0))$, the distribution of $X(.)$ can the be evaluated

Shift Idea - Observe a Path in $Y(.)$

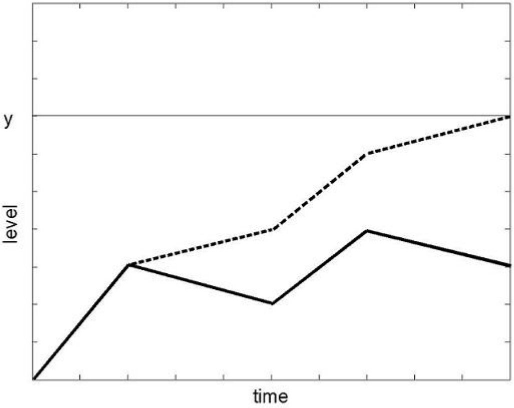


Figure: Start in $(i, 0)$, end in (j, y). Consider $Z(\hat{\theta}(y))$.

Corresponding Laplace-Stieltjes Transform

Definition

Let $\hat{\Delta}_X^y(s) = [\hat{\Delta}_X^y(s)_{ij}]$ *be such that for all* $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$

$$\hat{\Delta}_X^y(s)_{ij}$$

is given by

$$E(e^{-sx} : \varphi(\hat{\theta}(y)) = j, Z(t) = x | \varphi(0) = i, Y(t) = 0, X(t) = 0)$$

Matrix $\mathbf{W}(s)$

$$\text{Assume } s \text{ such that} \quad \max_{i:c_i>0} \frac{\mathbf{T}_{ii}}{c_i} < Re(s) < \min_{i:c_i<0} \frac{\mathbf{T}_{ii}}{c_i} \qquad (1)$$

$$\begin{aligned}
\mathbf{W}_{11}(s) &= \mathbf{R}_1^{-1}[(\mathbf{T}_{11} - s\mathbf{D}_1) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{01}] \\
\mathbf{W}_{22}(s) &= \mathbf{R}_2^{-1}[(\mathbf{T}_{22} - s\mathbf{D}_2) - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{02}] \\
\mathbf{W}_{12}(s) &= \mathbf{R}_1^{-1}[\mathbf{T}_{12} - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{02}] \\
\mathbf{W}_{21}(s) &= \mathbf{R}_2^{-1}[\mathbf{T}_{21} - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{D}_0)^{-1}\mathbf{T}_{01}]
\end{aligned}$$

where $\mathbf{D}_k = diag(c_i)_{i\in\mathcal{S}_k}$ for $k = 0, 1, 2$

Definition

$$\mathbf{W}(s) = \begin{bmatrix} \mathbf{W}_{11}(s) & \mathbf{W}_{12}(s) \\ \mathbf{W}_{21}(s) & \mathbf{W}_{22}(s) \end{bmatrix}$$

$\mathbf{W}(s)$ as the Generator of the 2-D FM

Theorem

$$\hat{\Delta}_X^y(s) = e^{\mathbf{W}(s)y}$$

Proof.

- Note that $\hat{\Delta}_X^{y+u}(s) = \hat{\Delta}_X^y(s)\hat{\Delta}_X^u(s)$ and $\lim_{y\to 0^+} \hat{\Delta}_X^y(s) = \mathbf{I}$
- Evaluate $\hat{\Delta}_X^h(s)$ for small h
- Show that $\lim_{h\to 0^+} \frac{\hat{\Delta}_X^h(s)-\mathbf{I}}{h} = \mathbf{W}(s)$

□

Shift Idea - Return to Level Zero in $Y(.)$

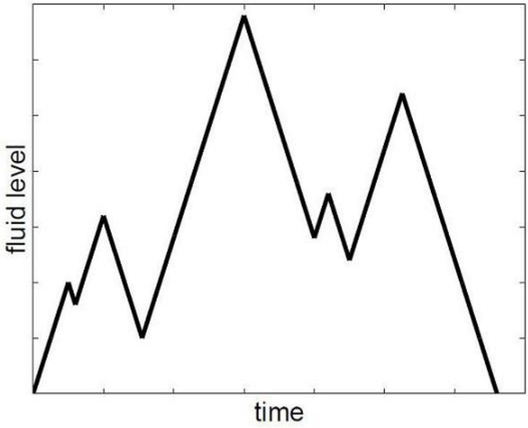


Figure: Start in $(i, 0)$, end in $(j, 0)$. Consider $Z(\theta(0))$.

Matrix $\mathbf{\Psi}_X(s)$

Definition

For s with $Re(s) \geq 0$, $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$, let $\Psi_X(s)_{ij}$ be given by

$$E[e^{-sZ(\theta(y))} : \theta(y) < \infty, \varphi(\theta(y)) = j | Y(0) = y, \varphi(0) = i]$$

Let

$$\mathbf{\Psi}_X(s) = [\Psi_X(s)_{ij}]$$

Riccati Equation for $\mathbf{\Psi}_X(s)$

Theorem

If s is real, then $\mathbf{\Psi}_X(s)$ is the minimal nonnegative solution of

$$\mathbf{W}_{12}(s)+\mathbf{\Psi}_X(s)\mathbf{W}_{21}(s)\mathbf{\Psi}_X(s)+\mathbf{W}_{11}(s)\mathbf{\Psi}_X(s)+\mathbf{\Psi}_X(s)\mathbf{W}_{22}(s)=0$$

Efficient Algorithm for $\mathbf{\Psi}_X(s)$

- Let $\mathbf{\Psi}_X(s,0) = \mathbf{0}$
- For $n \geq 1$, evaluate $\mathbf{\Psi}_X(s,n+1)$, by solving

$$\mathbf{A}\mathbf{\Psi}_X(s,n+1) + \mathbf{\Psi}_X(s,n+1)\mathbf{B} \quad = \quad \mathbf{C}$$

where

$$\begin{aligned}
\mathbf{A} &= \mathbf{W}_{11}(s) + \mathbf{\Psi}_X(s,n)\mathbf{W}_{21}(s) \\
\mathbf{B} &= \mathbf{W}_{22}(s) + \mathbf{W}_{21}(s)\mathbf{\Psi}_X(s,n) \\
\mathbf{C} &= -\mathbf{W}_{12}(s) + \mathbf{\Psi}_X(s,n)\mathbf{W}_{21}(s)\mathbf{\Psi}_X(s,n)
\end{aligned}$$

until a stopping criterion is met

(Transient) Results Derived Using $\mathbf{W}(s)$ and $\mathbf{\Psi}_X(s)$

- LST of the shift in $X(.)$ for the following paths in $Y(.)$
  - Return to the original level
  - Draining/Filling to some level
  - Return to the original level while avoiding some taboo level
  - Draining/Filling to some level while avoiding some taboo level
- Treatment of models with unbounded/bounded buffer Y
- Treatment of models with multi-layers in buffer Y, and with boundaries at which the behaviour changes

Visual explorations of 2-D FMs

- Bounded with $X(t) \geq 0,\ Y(t) \geq 0$ *
- Unbounded *
- Unbounded with no drift *

For more, check out **drMalgorzata** on youtube!

Upward Shift Idea: $\Psi(s)$ Revisited

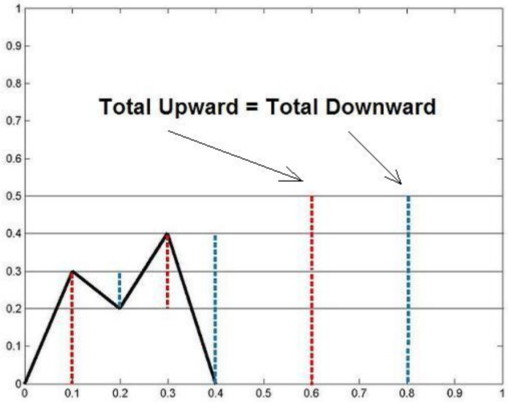


Figure: Upward shift $Z^{+}(\theta(0))$ = Downward shift $Z^{-}(\theta(0))$

Upward Shift Idea - Observe a Path in $Y(.)$

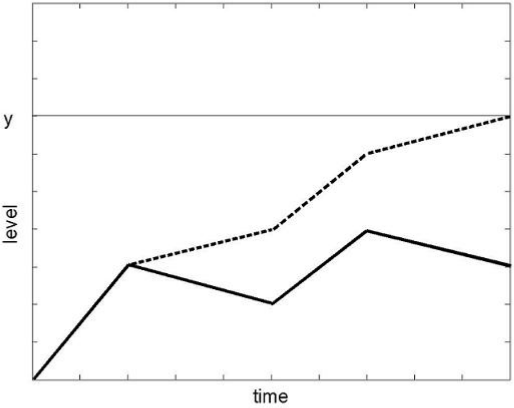


Figure: Start in $(i, 0)$, end in (j, y). Consider $Z^{+}(\hat{\theta}(y))$.

Matrix $\mathbf{Q}^+(s)$

Definition

For s with Re$(s) \geq 0$

$$\mathbf{Q}^+(s) = \begin{bmatrix} \mathbf{Q}_{11} - s\mathbf{I} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$$

$\mathbf{Q}^{+}(s)$ as a Generator

Let $Z^{+}(t)$ be the *total upward shift* in $Y(.)$ at time t, given by

$$Z^{+}(t) = \int_{u=0}^{t} r_{\varphi(u)} \times I(r_{\varphi(u)} > 0) du$$

Theorem

The LST of $Z^{+}(.)$ at time $\hat{\theta}(y)$,

$$E(e^{-sx} : \varphi(\hat{\theta}(y)) = j, Z^{+}(t) = x | \varphi(0) = i, Y(t) = 0)$$

is given by

$$[e^{\mathbf{Q}^{+}(s)y}]_{ij}$$

Matrix $\mathbf{M}$

Let

- $\mathbf{f}_y(x) = [f_y(x)_{ij}]$, $0 \leq x \leq y$, be the inverse of $e^{\mathbf{Q}_+(s)y}$
- $\mathbf{M} = [M_{ij}]$ for all $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$M_{ij} = \int_{x=0}^{\infty} f_{2x}(x)_{ij} dx$$

- $\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}$

Alternative Expression for $\mathbf{M}$

Theorem

Matrix $\mathbf{M}$ *is given by*

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\Psi}\mathbf{M}_{21} & (\mathbf{I} - \boldsymbol{\Psi}\boldsymbol{\Xi})^{-1}\boldsymbol{\Psi} \\ \boldsymbol{\Xi}(\mathbf{I} - \boldsymbol{\Psi}\boldsymbol{\Xi})^{-1} & \boldsymbol{\Xi}\mathbf{M}_{12} \end{bmatrix}\end{aligned}$$

New Riccati Equation for $\mathbf{\Psi}$

Theorem

$$\mathbf{\Psi} + \mathbf{\Psi}\mathbf{M}_{21}\mathbf{\Psi} = \mathbf{M}_{12}$$

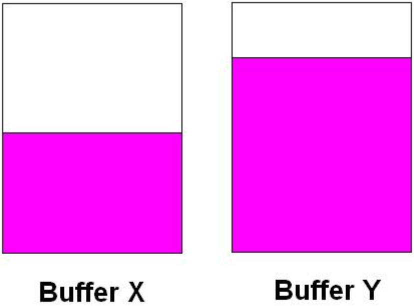
Compare with

$$\mathbf{Q}_{12} + \mathbf{Q}_{11}\mathbf{\Psi} + \mathbf{\Psi}\mathbf{Q}_{22} + \mathbf{\Psi}\mathbf{Q}_{21}\mathbf{\Psi} = \mathbf{0}$$

Definition of a Fluid-Fluid Model

Let $\{(\varphi(t), X(t), Y(t)), t \geq 0\}$ be a process such that

- $\{(\varphi(t), X(t)), t \geq 0\}$ is a 1-D FM with set of phases $\mathcal{S}$, generator $\mathbf{T}$ and rates c_i
- $\{(\varphi(t), X(t)), t \geq 0\}$ is the driving process
- $Y(t)$ is the level of the Fluid Model with rates $r_i(x)$
- $X(t) \in (-\infty, +\infty)$ or $X(t) \geq 0$
- $Y(t) \geq 0$



$$\frac{dX(t)}{dt} = c_i \quad \text{when} \quad \varphi(t) = i$$

$$\frac{dY(t)}{dt} = r_i(x) \quad \text{when} \quad \varphi(t) = i, X(t) = x \text{ and } Y(t) > 0$$

Example - Hydro-Power Generator

If the generator is newer, it may operate more efficiently, produce more energy and require less-costly maintenance.

To model this, let

- $\varphi(t) \in \{1, 2, 3, 4, 5, 6\}$ be the phase at time t
- $X(t) \in [0, 1]$ be the deterioration level, with rates c_i
- $Y(t)$ be the total revenue level, with rates $r_i(x)$

Analysis Overview: Operator-Analytic Methods

1. Derive the generator B of $\{(\varphi(t), X(t)), t \geq 0\}$

   with respect to time

2. Derive the generator D of the Fluid-Fluid Model

   with respect to the in-out fluid in the process $\{Y(t); t \geq 0\}$

Some Notation

- $\mathcal{F}$ = Borel-measurable set of all possible values of $X(t)$
- $\mathcal{F}^{(+)}(k) = \{u : r_k(u) > 0\}$ for given $k \in \mathcal{S}$
- $\mathcal{F}^{(-)}(k) = \{u : r_k(u) < 0\}$ for given $k \in \mathcal{S}$
- $\mathcal{F}^{(0)}(k) = \{u : r_k(u) = 0\}$ for given $k \in \mathcal{S}$
- $\mathcal{S}_+ = \{i \in \mathcal{S} : \mathcal{F}^{(+)}(i) \neq \emptyset\}$
- $\mathcal{S}_- = \{i \in \mathcal{S} : \mathcal{F}^{(-)}(i) \neq \emptyset\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : \mathcal{F}^{(0)}(i) \neq \emptyset\}$

Definition of Operator V

Define the matrix of operators

$$V(t) = [V_{ij}^{\ell m}(t)]_{i \in \mathcal{S}_\ell, j \in \mathcal{S}_m; \ell, m \in \{+,-,0\}}$$

such that

$$\mu_i^\ell V_{ij}^{\ell m}(t)(\mathcal{A})$$

is given by

$$\int_{x \in \mathcal{F}^{(\ell)}(i)} d\mu_i^\ell(x) P[\varphi(t) = j, X(t) \in \mathcal{A} | \varphi(0) = i, X(0) = x]$$

Interpretation of Operator V

$$\mu_i^\ell V_{ij}^{\ell m}(t)(\mathcal{A})$$

is the total probability of the process $\{(\varphi(t), X(t)), t \geq 0\}$

being in the destination set $(j, \mathcal{A})$ at time t,

assuming that it starts at time zero in the set $(i, \mathcal{F}^{(\ell)}(i))$

according to the measure μ_i^ℓ

Expression for Generator B

$$V(t) = e^{Bt}$$

$B = [B_{ij}^{\ell m}]_{i \in \mathcal{S}_\ell, j \in \mathcal{S}_m, \ell, m \in \{+,-,0\}}$

Case 1) for all $\ell \in \{+,-,0\}$ and $i \in \mathcal{S}_\ell$, $i \neq j$,

$$\mu_i^\ell B_{ij}^{\ell m}(\mathcal{A}) = T_{ij} \mu_i^\ell (\mathcal{A} \cap \mathcal{F}^{(\ell)}(i))$$

Case 2) for all $\ell \in \{+,-,0\}$, $\ell \neq m$,

$$\begin{aligned} \mu_j^\ell B_{jj}^{\ell m}(\mathcal{A}) = \; & I(c_j > 0) c_j \nu_j^\ell(u) I(u \neq v) I(u \in \partial_{R \setminus L} \left(\overline{\mathcal{F}^{(\ell)}(j)}\right)) \\ & - I(c_j < 0) c_j \nu_j^\ell(v) I(u \neq v) I(v \in \partial_{L \setminus R} \left(\overline{\mathcal{F}^{(\ell)}(j)}\right)) \\ & - I(c_j < 0) c_j \nu_j^\ell(0) I(v = 0) I(0 \in \partial_{L \setminus R}(\overline{\mathcal{F}^{(\ell)}(j)})) \end{aligned}$$

Case 3) otherwise

$$\begin{aligned}\mu_j^m B_{jj}^{mm}(\mathcal{A}) &= T_{jj}\mu_j^m(\mathcal{A})\\ &+I(c_j > 0)I(u \neq v)\left[c_j\nu_j^m(u)I(u \notin \partial_L\left(\overline{\mathcal{F}^{(m)}(j)}\right)) - c_j\nu_j^m(v)\right]\\ &+I(c_j < 0)I(u \neq v)\left[c_j\nu_j^m(u) - c_j\nu_j^m(v)I(v \notin \partial_R\left(\overline{\mathcal{F}^{(m)}(j)}\right))\right]\\ &-I(c_j < 0)c_j\nu_j^\ell(0)I(v = 0)I(0 \in \partial_{L\setminus R}(\overline{\mathcal{F}^{(m)}(j)}))\end{aligned}$$

Definition of Operator U

Let $U(y,s) = [U_{ij}^{\ell m}(y,s)]_{i\in\mathcal{S}_\ell, j\in\mathcal{S}_m; \ell,m\in\{+,-\}}$ be such that

$$U_{ij}^{\ell m}(y,s)$$

is given by

$$\int_{x\in\mathcal{F}^{(\ell)}(i)} d\mu_i^\ell(x) E[e^{-s\omega(y)}$$

$$\times I(\varphi(\omega(y)) = j, X(\omega(y)) \in \mathcal{A}) | \varphi(0) = i, X(0) = x]$$

Interpretation of Operator U

$$\mu_i^\ell U_{ij}^{\ell m}(y, s)(\mathcal{A})$$

is the LST of the time taken for the total amount of fluid that has flowed into or out of the buffer to reach y

and do so with the process $\{(\varphi(t), X(t)), t \geq 0\}$ in the destination set $(j, \mathcal{A})$

assuming the process starts in $(i, \mathcal{F}^{(\ell)}(i))$ at time zero according to the measure μ_i^ℓ

Expression for Operator D

$$U(y,s) = e^{D(s)y}$$

where $D(s) = \left[D_{ij}^{\ell m}(s)\right]_{i\in\mathcal{S}_\ell, j\in\mathcal{S}_m; \ell,m\in\{+,-\}}$ and

$$D_{ij}^{\ell m}(s) = \left[R^{(\ell)}\left(B^{(\ell m)} - sI + B^{(\ell 0)}(sI - B^{(00)})^{-1}B^{(0m)}\right)\right]_{ij}$$

where $R^{(\ell)} = \mathrm{diag}(R_i^{(\ell)})_{i\in\mathcal{S}_\ell}$ is a diagonal matrix of operators such that

$$R_i^{(\ell)}(x,\mathcal{A}) = \frac{1}{|r_i(x)|}I(x\in\mathcal{A})$$

Results

Theretical framework:

- Transient
- Stationary

Current work:

- Numerical solutions for expressions involving operator
- Discretization of a Fluid Model that preserves its important statistical properties

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Thanks for listening!