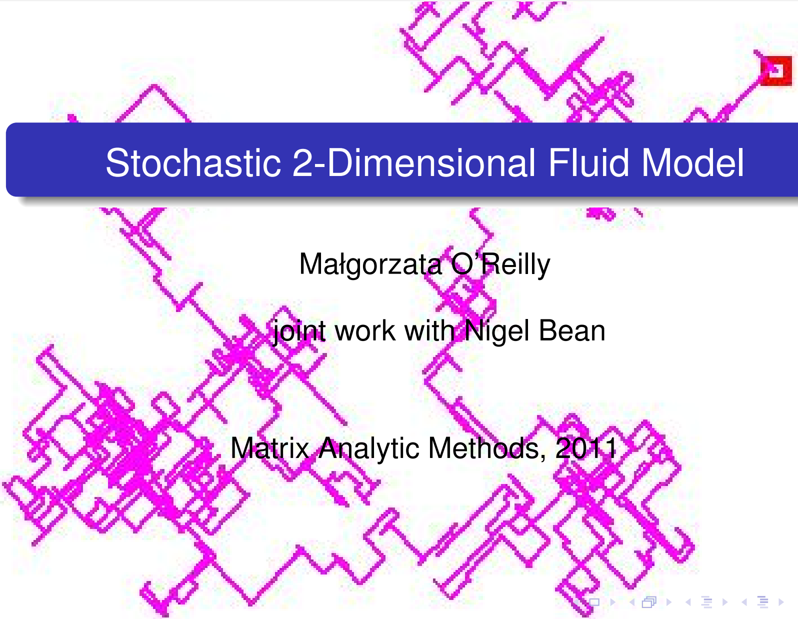

Stochastic 2-Dimensional Fluid Model

Małgorzata O'Reilly

joint work with Nigel Bean

Matrix Analytic Methods, 2011

Outline

1. Construction of the Model
   - Continuous-time Markov Chain
   - 1-Dimensional Fluid Model (earlier work)
   - 2-Dimensional Fluid Model
2. Results for 2-D FMs
   - Main Results
   - New Results for 1-D FM
   - More General Scenario
3. Current/Future Work

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CTMC and its Generator Matrix **T**

Let $\{\varphi(t), t \geq 0\}$ be a CTMC with:

- $\mathcal{S}$ - (finite) set of all possible *phases* $\varphi(t)$
- **T** - (irreducible) generator matrix

Fact

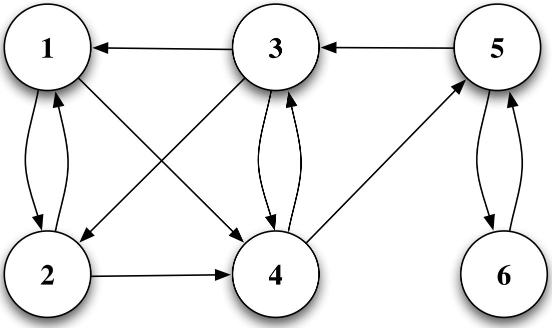
Let $\mathbf{P}(t) = [P(t)_{ij}]$ *be a matrix such that*

$$P(t)_{ij} = P(\varphi(t) = j | \varphi(0) = i)$$

Then

$$\mathbf{P}(t) = e^{\mathbf{T}t}$$

Example - Hydro-Power Generator



1 on-design, 2 off-design, 3 start, 4 stop, 5 idle, 6 maintenance

Stationary Distribution Vector π

Fact

$\boldsymbol{\pi} = [\pi_i]_{i \in \mathcal{S}}$ *is the solution of*

$$\begin{cases} \pi \mathbf{T} & = & \mathbf{0} \\ \pi \mathbf{1} & = & 1 \end{cases}$$

where $\mathbf{0}$ *is a row vector of zeros,* $\mathbf{1}$ *is a column vector of ones*

Example - Hydro-Power Generator

Given

- CTMC with $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ and generator $\mathbf{T}$
- Revenue rate c_i for all $i \in \mathcal{S}$

we can derive

$$\text{long-run mean revenue} = \sum_i \pi_i c_i$$

But we would like to do better than this!

Outline

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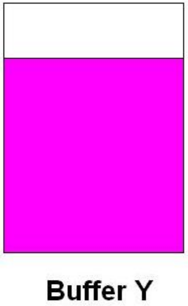
Definition of a 1-D FM

Let $\{(\varphi(t), Y(t)), t \geq 0\}$ be a process such that:

- $\{\varphi(t), t \geq 0\}$ is a CTMC with the set of phases $\mathcal{S}$ and generator $\mathbf{T}$
- *Level* $Y(t)$ records some performance measure
- When $\varphi(t) = i$, the rate at which $Y(t)$ is changing is r_i



$$\frac{dY(t)}{dt} = r_i \quad \text{when} \quad \varphi(t) = i \ \text{ and } \ Y(t) > 0$$

Example - Hydro-Power Generator

To model the deterioration process, let

- $Y(t) \in [0, 1]$ be the deterioration level
- 0 - brand new, 1 - needs replacement
- r_i be deterioration rates, $i \in \{1, 2, 3, 4, 5, 6\}$

In-Out Fluid Idea

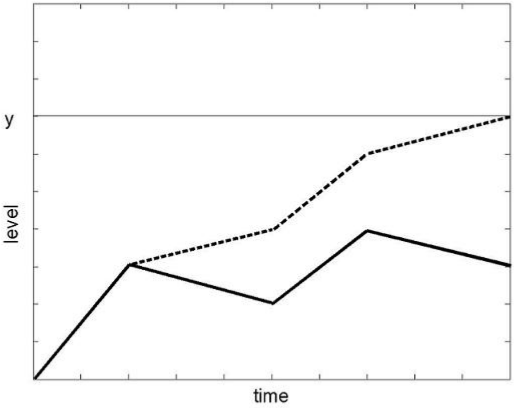


Figure: Start in $(i, 0)$, end in (j, y) at time $\hat{\theta}(y)$

Some Notation

- $\mathcal{S}_1 = \{i \in \mathcal{S} : r_i > 0\}$
- $\mathcal{S}_2 = \{i \in \mathcal{S} : r_i < 0\}$
- $\mathcal{S}_0 = \{i \in \mathcal{S} : r_i = 0\}$
- $\mathbf{R}_1 = diag(r_i)$ for all $i \in \mathcal{S}_1$
- $\mathbf{R}_2 = diag(|r_i|)$ for all $i \in \mathcal{S}_2$
- $\mathbf{T}_{11} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_1$
- $\mathbf{T}_{12} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_1, j \in \mathcal{S}_2$
- $\mathbf{T}_{21} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_2, j \in \mathcal{S}_1$
- $\mathbf{T}_{22} = [\mathbf{T}_{ij}]$ for all $i \in \mathcal{S}_2, j \in \mathcal{S}_2$

Matrix $\mathbf{Q}(s)$

$$\begin{aligned}
\mathbf{Q}_{11}(s) &= \mathbf{R}_1^{-1}[(\mathbf{T}_{11} - s\mathbf{I}) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}] \\
\mathbf{Q}_{22}(s) &= \mathbf{R}_2^{-1}[(\mathbf{T}_{22} - s\mathbf{I}) - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}] \\
\mathbf{Q}_{12}(s) &= \mathbf{R}_1^{-1}[\mathbf{T}_{12} - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{02}] \\
\mathbf{Q}_{21}(s) &= \mathbf{R}_2^{-1}[\mathbf{T}_{21} - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{I})^{-1}\mathbf{T}_{01}]
\end{aligned}$$

Definition

For s with $Re(s) \geq 0$

$$\mathbf{Q}(s) = \begin{bmatrix} \mathbf{Q}_{11}(s) & \mathbf{Q}_{12}(s) \\ \mathbf{Q}_{21}(s) & \mathbf{Q}_{22}(s) \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{Q}(0) = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$$

$\mathbf{Q}(s)$ as the Generator of the 1-D FM

Let

- $|Y(t)| = \int_0^t |r_{\varphi(u)}| du$
- $\hat{\theta}(y) = \inf\{t \geq 0 : |Y(t)| = y\}$

Theorem

$$E(e^{-s\hat{\theta}(y)} : \varphi(\hat{\theta}(y)) = j | \varphi(0) = i, Y(t) = 0)$$

is given by

$$[e^{\mathbf{Q}(s)y}]_{ij}$$

Return to Level Zero in $Y(.)$

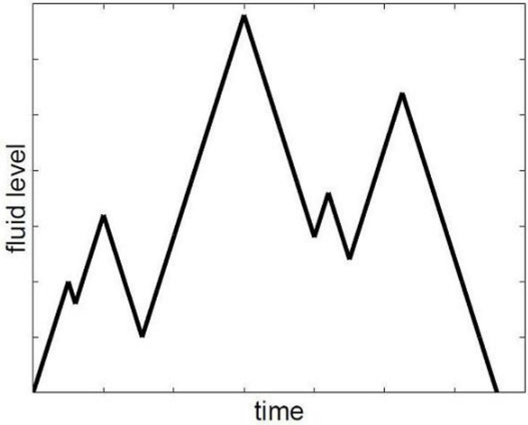


Figure: Start in $(i, 0)$, end in $(j, 0)$ at time $\theta(0)$

Matrix $\mathbf{\Psi}(s)$

Let $\theta(0) = \inf\{t \geq 0 : Y(t) = 0\}$

Definition

For s with $Re(s) \geq 0$, i with $r_i > 0$, j with $r_j < 0$, let

$$\Psi(s)_{ij} = E(\theta(0) < \infty, \theta(0) = i | \varphi(0) = i, Y(0) = 0)$$

Let

$$\mathbf{\Psi}(s) = [\Psi(s)_{ij}]$$

$$\mathbf{\Psi} = \mathbf{\Psi}(0) = [\Psi_{ij}]$$

Riccati Equation for $\boldsymbol{\Psi}$

Fact

For $s \geq 0$, $\boldsymbol{\Psi}(s)$ is the minimum nonnegative solution of

$$\mathbf{Q}_{12}(s) + \mathbf{Q}_{11}(s)\boldsymbol{\Psi}(s) + \boldsymbol{\Psi}(s)\mathbf{Q}_{22}(s) + \boldsymbol{\Psi}(s)\mathbf{Q}_{21}(s)\boldsymbol{\Psi}(s) = \mathbf{0}$$

Outline

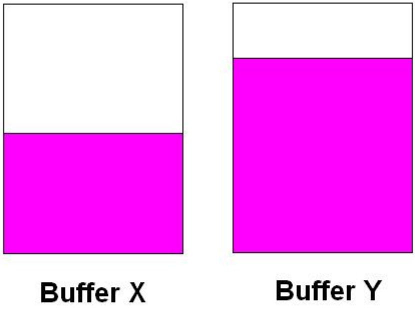
Definition of a 2-D FM

Let $\{(\varphi(t), X(t), Y(t)), t \geq 0\}$ be a process such that

- $\{(\varphi(t), Y(t)), t \geq 0\}$ is a 1-D FM with the set of phases $\mathcal{S}$, generator $\mathbf{T}$ and rates r_i
- $\{(\varphi(t), X(t)), t \geq 0\}$ is a 1-D FM with set of phases $\mathcal{S}$, generator $\mathbf{T}$ and rates c_i

We study the case

- $X(t) \in (-\infty, +\infty)$
- $Y(t) \geq 0$



$$\frac{dX(t)}{dt} = c_i \quad \text{when} \quad \varphi(t) = i$$

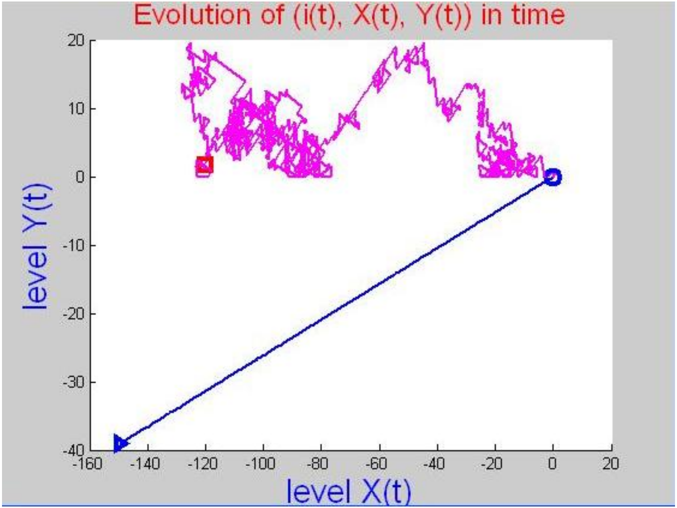
$$\frac{dY(t)}{dt} = r_i \quad \text{when} \quad \varphi(t) = i \ \text{ and } \ Y(t) > 0$$

Example - Hydro-Power Generator

To model the deterioration and revenue processes, let

- $\varphi(t) \in \{1, 2, 3, 4, 5, 6\}$ be the phase at time t
- $Y(t) \in [0, 1]$ be the deterioration level, with rates r_i
- $X(t) \in (-\infty, +\infty)$ be the total revenue level, with rates c_i

Sample Path Example - $X(t) \in (-\infty, +\infty)$, $Y(t) \geq 0$



*

Outline

Shift Idea

- Let $Z(t) = X(t) - X(0)$ be the total *shift* in $X(.)$ at time t
- Evaluate the Laplace-Stieltjes transform of shift $Z(.)$ for a path in $Y(.)$
- Then, given initial state $(\varphi(0), X(0), Y(0))$, the distribution of $X(.)$ can the be evaluated

Shift Idea - Observe a Path in $Y(.)$

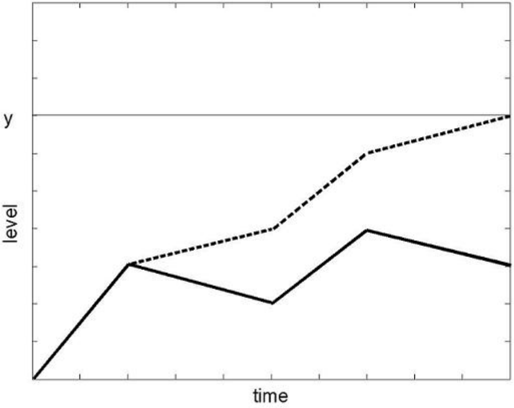


Figure: Start in $(i, 0)$, end in (j, y). Consider $Z(\hat{\theta}(y))$.

Matrix $\mathbf{W}(s)$

Assume s such that $\max_{i:c_i>0} \frac{\mathbf{T}_{ii}}{c_i} < Re(s) < \min_{i:c_i<0} \frac{\mathbf{T}_{ii}}{c_i}$ (1)

$$
\begin{aligned}
\mathbf{W}_{11}(s) &= \mathbf{R}_1^{-1}[(\mathbf{T}_{11} - s\mathbf{C}_{1\bullet}) - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{C}_{0\bullet})^{-1}\mathbf{T}_{01}] \\
\mathbf{W}_{22}(s) &= \mathbf{R}_2^{-1}[(\mathbf{T}_{22} - s\mathbf{C}_{2\bullet}) - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{C}_{0\bullet})^{-1}\mathbf{T}_{02}] \\
\mathbf{W}_{12}(s) &= \mathbf{R}_1^{-1}[\mathbf{T}_{12} - \mathbf{T}_{10}(\mathbf{T}_{00} - s\mathbf{C}_{0\bullet})^{-1}\mathbf{T}_{02}] \\
\mathbf{W}_{21}(s) &= \mathbf{R}_2^{-1}[\mathbf{T}_{21} - \mathbf{T}_{20}(\mathbf{T}_{00} - s\mathbf{C}_{0\bullet})^{-1}\mathbf{T}_{01}]
\end{aligned}
$$

where $\mathbf{C}_{k\bullet} = diag(c_i)_{i\in\mathcal{S}_k}$ for $k = 0, 1, 2$

Definition

$$
\mathbf{W}(s) = \begin{bmatrix} \mathbf{W}_{11}(s) & \mathbf{W}_{12}(s) \\ \mathbf{W}_{21}(s) & \mathbf{W}_{22}(s) \end{bmatrix}
$$

$\mathbf{W}(s)$ as the Generator of the 2-D FM

Theorem

The Laplace-Stieltjes transform of $Z(.)$ at time $\hat{\theta}(y)$,

$$E(e^{-sx} : \varphi(\hat{\theta}(y)) = j, Z(t) = x | \varphi(0) = i, Y(t) = 0, X(t) = 0),$$

is given by

$$[e^{\mathbf{W}(s)y}]_{ij}$$

Shift Idea - Return to Level Zero in $Y(.)$

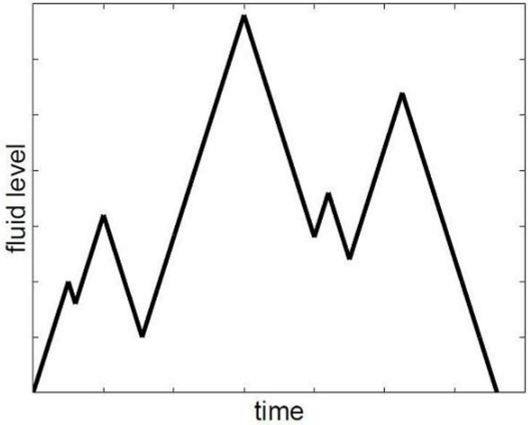


Figure: Start in $(i, 0)$, end in $(j, 0)$. Consider $Z(\theta(0))$.

Matrix $\boldsymbol{\Psi}_X(s)$

Definition

For s with $Re(s) \geq 0$, $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$, *let* $\Psi_X(s)_{ij}$ *be given by*

$$E[e^{-sZ(\theta(y))} : \theta(y) < \infty, \varphi(\theta(y)) = j | Y(0) = y, \varphi(0) = i]$$

Let

$$\boldsymbol{\Psi}_X(s) = [\Psi_X(s)_{ij}]$$

Riccati Equation for $\mathbf{\Psi}_X(s)$

Theorem

If s is real, then $\mathbf{\Psi}_X(s)$ *is the minimal nonnegative solution of*

$$\mathbf{W}_{12}(s)+\mathbf{\Psi}_X(s)\mathbf{W}_{21}(s)\mathbf{\Psi}_X(s)+\mathbf{W}_{11}(s)\mathbf{\Psi}_X(s)+\mathbf{\Psi}_X(s)\mathbf{W}_{22}(s)=0$$

Efficient Algorithm for $\mathbf{\Psi}_X(s)$ (one of three)

- Let $\mathbf{\Psi}_X(s,0) = \mathbf{0}$
- For $n \geq 1$, evaluate $\mathbf{\Psi}_X(s,n+1)$, by solving

$$\mathbf{A}\mathbf{\Psi}_X(s,n+1) + \mathbf{\Psi}_X(s,n+1)\mathbf{B} = \mathbf{C}$$

where

$$\begin{aligned} \mathbf{A} &= \mathbf{W}_{11}(s) + \mathbf{\Psi}_X(s,n)\mathbf{W}_{21}(s) \\ \mathbf{B} &= \mathbf{W}_{22}(s) + \mathbf{W}_{21}(s)\mathbf{\Psi}_X(s,n) \\ \mathbf{C} &= -\mathbf{W}_{12}(s) + \mathbf{\Psi}_X(s,n)\mathbf{W}_{21}(s)\mathbf{\Psi}_X(s,n) \end{aligned}$$

until a stopping criterion is met

Results Derived Using $\mathbf{W}(s)$ and $\mathbf{\Psi}_X(s)$

- Laplace-Stieltjes transform of the shift in $X(.)$ for the following paths in $Y(.)$
  - Return to the original level
  - Draining/Filling to some level
  - Return to the original level while avoiding some taboo level
  - Draining/Filling to some level while avoiding some taboo level
- Treatment of models with unbounded/bounded buffer Y
- Treatment of models with multi-layers in buffer Y, and with boundaries at which the behaviour changes

Outline

Upward Shift Idea

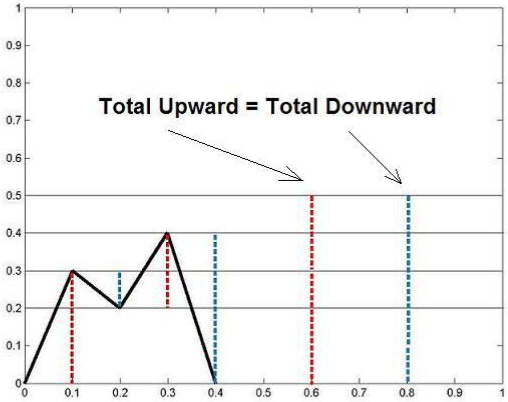


Figure: Upward shift $Z^{+}(\theta(0))$ = Downward shift $Z^{-}(\theta(0))$

Upward Shift Idea - Observe a Path in $Y(.)$

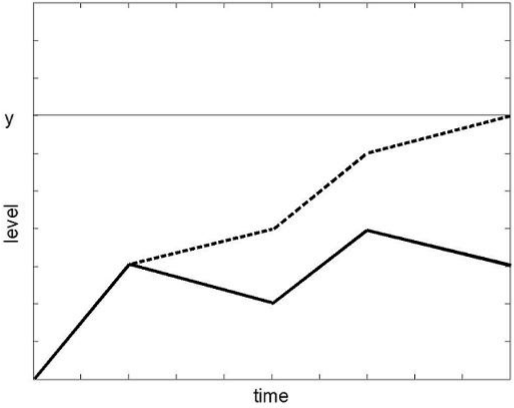


Figure: Start in $(i, 0)$, end in (j, y). Consider $Z^{+}(\hat{\theta}(y))$.

Matrix $\mathbf{Q}^{+}(s)$

Definition

For s with Re(s) ≥ 0

$$\mathbf{Q}^{+}(s) = \begin{bmatrix} \mathbf{Q}_{11} - s\mathbf{I} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}$$

$\mathbf{Q}^+(s)$ as a Generator

Let $Z^+(t)$ be the *total upward shift* in $X(.)$ at time t, given by

$$Z^+(t) = \int_0^t c_{\varphi(u)} \times I(c_{\varphi(u)} > 0) du$$

Theorem

The Laplace-Stieltjes transform of $Z^+(.)$ at time $\hat{\theta}(y)$,

$$E(e^{-sx} : \varphi(\hat{\theta}(y)) = j, Z^+(t) = x | \varphi(0) = i, Y(t) = 0, X(t) = 0)$$

is given by

$$[e^{\mathbf{Q}^+(s)y}]_{ij}$$

Matrix **M**

Let

- $\mathbf{f}_y(x) = [f_y(x)_{ij}]$, $0 \le x \le y$, be the inverse of $e^{\mathbf{Q}_+(s)y}$
- $\mathbf{M} = [M_{ij}]$ for all $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$, where

$$M_{ij} = \int_0^\infty f_{2x}(x)_{ij} dx$$

- $\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}$

Alternative Expression for **M**

Theorem

Matrix **M** *is given by*

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}\mathbf{M}_{21} & (\mathbf{I} - \mathbf{\Psi}\mathbf{\Xi})^{-1}\mathbf{\Psi} \\ \mathbf{\Xi}(\mathbf{I} - \mathbf{\Psi}\mathbf{\Xi})^{-1} & \mathbf{\Xi}\mathbf{M}_{12} \end{bmatrix}$$

New Riccati Equation for $\boldsymbol{\Psi}$

Theorem

$$\boldsymbol{\Psi} + \boldsymbol{\Psi}\mathbf{M}_{21}\boldsymbol{\Psi} = \mathbf{M}_{12}$$

Compare with

$$\mathbf{Q}_{12} + \mathbf{Q}_{11}\boldsymbol{\Psi} + \boldsymbol{\Psi}\mathbf{Q}_{22} + \boldsymbol{\Psi}\mathbf{Q}_{21}\boldsymbol{\Psi} = \mathbf{0}$$

New Riccati Equation for $\boldsymbol{\Psi}$

Theorem

$$\boldsymbol{\Psi} + \boldsymbol{\Psi}\mathbf{M}_{21}\boldsymbol{\Psi} = \mathbf{M}_{12}$$

Compare with

$$\mathbf{Q}_{12} + \mathbf{Q}_{11}\boldsymbol{\Psi} + \boldsymbol{\Psi}\mathbf{Q}_{22} + \boldsymbol{\Psi}\mathbf{Q}_{21}\boldsymbol{\Psi} = \mathbf{0}$$

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More General Scenario

- Rates $r_i(x)$ at which $Y(.)$ is changing depend on i and x

The results involve a generator which is

- a matrix of operators

Details - the talk about the **Fluid-Fluid Model** by Nigel

Technique Used in the Analysis of Fluid Models

- Consider $Y(.)$ changing from y to $y + h$ for small $h > 0$
- Evaluate the corresponding Laplace-Stieltjes transform of interest
- Evaluate the Generator by taking appropriate limits with respect to level

Current/Future Work

- Investigating the new Riccati equation for $\boldsymbol{\Psi}$
- Bounded 2-D FMs for a tandem *
- Numerical methods for the operator expressions: with Zbyszek Palmowski from Wrocław University, Poland
- Multi-D FMs for networks
- Visual exploration of Multi-D FMs * * *
- Applications

Thanks for listening!