Asymptotic periodic analysis of cyclic stochastic fluid flows with time-varying transition rates

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Outline

Motivation

- We are interested in Stochastic Fluid Models (SFMs) with *time-varying parameters* so that we can model a wider range of problems.
- We consider a Cyclic Stochastic Fluid Model (CSFM) analysed in Margolius and O'Reilly (2016).
- We extend the analysis to the study of its asymptotic periodic distribution.
- We also study some other CSFMs of interest, with boundaries.

Cyclic Stochastic Fluid Model (CSFM)

- CSFM $\{(\widehat{X}(t), J(t)) : t \geq 0\}$ is a process with level variable $X(t) \geq 0$ and phase variable $J(t) \in \mathcal{S}$
- $\{J(t) : t \geq 0\}$ is a continuous-time Markov chain with
- time-varying generator $T(t) = [\mathcal{T}(t)_{ij}]_{i,j \in \mathcal{S}}$
- cycle of length 1 such that $T(t) = T(t+1)$ for all $t \geq 0$
- some finite irreducible state space $\mathcal{S}$
- fluid rates in $c_i \in \mathbb{R}$, for all $i \in \mathcal{S}$, such that

$$\frac{d}{dt}X(t) = c_{J(t)} \qquad \text{when } X(t) > 0,$$
$$\frac{d}{dt}X(t) = \max(0, c_{J(t)}) \qquad \text{when } X(t) = 0.$$

Example (Margolius and O'Reilly, 2016)

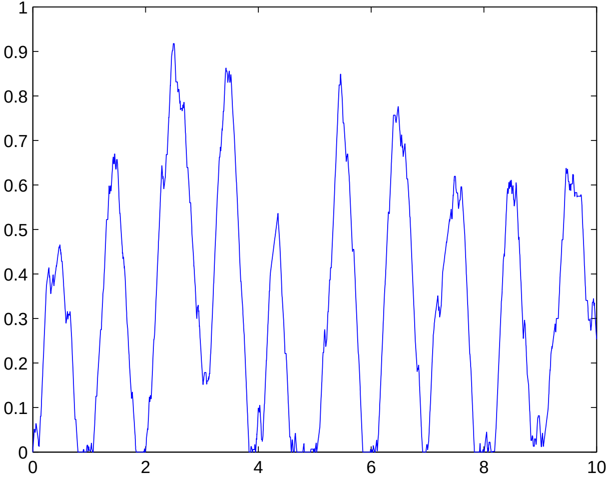
Let

$$T(t) = \begin{bmatrix} * & 0 & a_2(t) & 0 & a_3(t) & 0 & 0 & b_1(t) \\ 0 & * & a_1(t) & 0 & 0 & a_3(t) & 0 & b_2(t) \\ b_2(t) & b_1(t) & * & 0 & 0 & 0 & a_3(t) & 0 \\ 0 & 0 & 0 & * & a_1(t) & a_2(t) & 0 & b_3(t) \\ b_3(t) & 0 & 0 & b_1(t) & * & 0 & a_2(t) & 0 \\ 0 & b_3(t) & 0 & b_2(t) & 0 & * & a_1(t) & 0 \\ 0 & 0 & b_3(t) & 0 & b_2(t) & b_1(t) & * & 0 \\ a_1(t) & a_2(t) & 0 & 0 & a_3(t) & 0 & 0 & * \end{bmatrix}$$

where

$$\begin{aligned} a_1(t) &= 30 + 25\sin(2\pi t) \\ b_1(t) &= 30 - 20\sin(2\pi t) \\ a_2(t) &= 10 + 7\sin(2\pi t) \\ b_2(t) &= 40 - 35\sin(2\pi t) \\ a_3(t) &= 50 - 35\sin(2\pi t) \\ b_3(t) &= 80 + 50\sin(2\pi t). \end{aligned}$$

Evolution of level $X(t)$ in time (simulation)



Partitioning

Let

$$\begin{aligned} \mathcal{S}_+ &= \{i \in \mathcal{S} : c_i > 0\} \\ \mathcal{S}_- &= \{i \in \mathcal{S} : c_i < 0\} \\ \mathcal{S}_0 &= \{i \in \mathcal{S} : c_i = 0\} \end{aligned}$$

and partition according to $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_0$, as

$$T(t) = \begin{bmatrix} T_{++}(t) & T_{+-}(t) & T_{+0}(t) \\ T_{-+}(t) & T_{--}(t) & T_{-0}(t) \\ T_{0+}(t) & T_{00}(t) & T_{0-}(t) \end{bmatrix}.$$

Let

$$\mathbf{C}_+ = diag(c_i)_{i \in \mathcal{S}_+}, \quad \mathbf{C}_- = diag(c_i)_{i \in \mathcal{S}_-}.$$

Density $\pi(t; x)$ at $x > 0$

For , $t \in [0, 1)$, $x > 0$,

$$\pi(t; x) = \left[\ \pi(t; x)_{+} \quad \pi(t; x)_{-} \quad \pi(t; x)_{0}\ \right],$$

such that for all $t \in [0, 1)$ and $i \in \mathcal{S}$,

$$\pi(t; x)_i = \frac{\partial}{\partial x} \sum_{n=0}^{\infty} P(X(t+n) \leq x, J(t+n) = i).$$

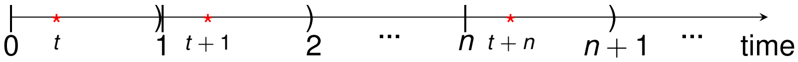
Mass $p(t)$ at $x = 0$

For $t \in [0, 1)$,

$$p(t) = \begin{bmatrix} \mathbf{0} & p_-(t) & p_0(t) \end{bmatrix},$$

where, for all $i \in \mathcal{S}_- \cup \mathcal{S}_0$,

$$p(t)_i = \sum_{n=0}^{\infty} P(X(t+n) = 0, J(t+n) = i).$$



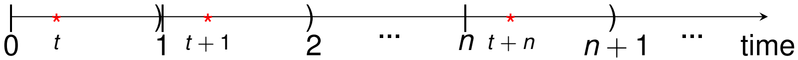
Matrix $\hat{U}_0(s,t)$

For all s, t with $0 \leq s \leq t$, define $U_0(s,t)$ which solves

$$\begin{aligned}
\frac{\partial}{\partial t} U_0(s,t) &= U_0(s,t) T_{00}(t) \\
\frac{\partial}{\partial s} U_0(s,t) &= -T_{00}(s) U_0(s,t) \\
U_0(s,s) &= \mathbf{I}.
\end{aligned}$$

where $U_0(s,t) = \mathbf{0}$ for $t < s$, and

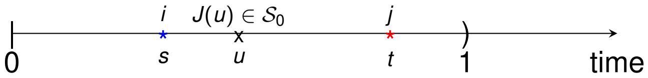
$$\hat{U}_0(s,t) = \sum_{n=0}^{\infty} U_0(s, t+n).$$



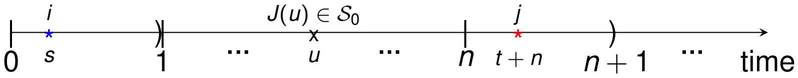
Physical interpretation of $\hat{U}_0(s,t)$

For all $i, j \in \mathcal{S}_0$,

$$[U_0(s,t)]_{ij} = P[J(t) = j \mid J(s) = i, J(u) \in \mathcal{S}_0, s \le u \le t]$$



$$[\hat{U}_0(s,t)]_{ij} = \sum_{n=0}^{\infty} P[J(t+n) = j \mid J(s) = i, J(u) \in \mathcal{S}_0, s \le u \le t+n]$$



Key fluid generator $\hat{Q}(s,t)$

For all s, t, with $0 \le s, t \le 1$, define

$$\hat{Q}(s,t) = \begin{bmatrix} \hat{Q}_{++}(s,t) & \hat{Q}_{+-}(s,t) \\ \hat{Q}_{-+}(s,t) & \hat{Q}_{--}(s,t) \end{bmatrix},$$

with

$$\begin{aligned}
\hat{Q}_{++}(s,t) &= \mathbf{C}_{+}^{-1}[T_{++}(s)I(s=t) + T_{+0}(s)\hat{U}_0(s,t)T_{0+}(t)] \\
\hat{Q}_{--}(s,t) &= \mathbf{C}_{-}^{-1}[T_{--}(s)I(s=t) + T_{-0}(s)\hat{U}_0(s,t)T_{0-}(t)] \\
\hat{Q}_{+-}(s,t) &= \mathbf{C}_{+}^{-1}[T_{+-}(s)I(s=t) + T_{+0}(s)\hat{U}_0(s,t)T_{0-}(t)] \\
\hat{Q}_{-+}(s,t) &= \mathbf{C}_{-}^{-1}[T_{-+}(s)I(s=t) + T_{-0}(s)\hat{U}_0(s,t)T_{0+}(t)].
\end{aligned}$$

Key matrix $\hat{\Psi}(s,t)$

For all s,t with $0 \le s \le t$, define matrix $\Psi(s,t) = [\Psi(s,t)_{ij}]_{i\in\mathcal{S}_+, j\in\mathcal{S}_-}$ such that, with $\theta_s(x) = \inf\{t > s : X(t) = x\}$, for all $i \in \mathcal{S}_+$, $j \in \mathcal{S}_-$,

$$\Psi(s,t)_{ij}dt \quad = \quad P\left[t < \theta_s(z) \le t + dt, J(\theta_s(z)) = j \;\middle|\; X(s) = z, J(s) = i\right],$$

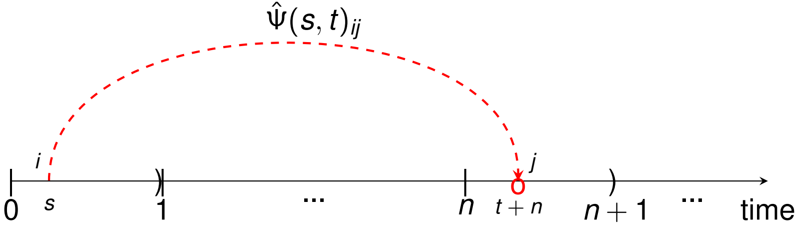
and matrix $\hat{\Psi}(s,t) = [\hat{\Psi}(s,t)_{ij}]$ such that

$$\hat{\Psi}(s,t) \quad = \quad \sum_{n=0}^{\infty} \Psi(s, t+n).$$

Physical interpretation of $\hat{\Psi}(s, t)$

$\hat{\Psi}(s, t)_{ij}$ is the probability that

- given start from level 0 in phase $i \in \mathcal{S}_+$ at time s,
- the process $\{(\widehat{X}(t), J(t)) : t \geq 0\}$ first returns to level 0
- at some time $u \in \bigcup_{n=0}^{\infty}(t + n, t + n + dt]$,
- and does so in phase $j \in \mathcal{S}_-$.



Expression for $\hat{\Psi}(s,t)$ (Margolius and O'Reilly, 2016)

By conditioning on level y (highest peak / lowest trough)

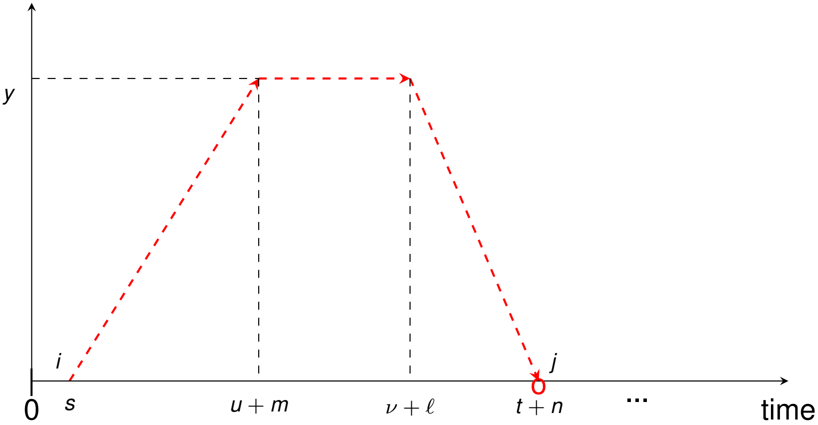
$$\begin{aligned}\hat{\Psi}(s,t) &= \int_{y=0}^{\infty}\int_{\nu=0}^{1}\int_{u=0}^{1} e^{\hat{A}(s,u)y} \\ &\times \left(\hat{Q}_{+-}(u,\nu) - \int_{\eta=0}^{1}\int_{\theta=0}^{1} \hat{\Psi}(u,\eta)\hat{Q}_{-+}(\eta,\theta)\hat{\Psi}(\theta,\nu)d\theta d\eta\right) \\ &\times e^{\hat{B}(\nu,t)y} du d\nu dy,\end{aligned}$$

where

$$\begin{aligned}\hat{A}(s,u) &= \hat{Q}_{++}(s,u) + \int_{z=0}^{1} \hat{\Psi}(s,z)\hat{Q}_{-+}(z,t)du, \\ \hat{B}(\nu,t) &= \hat{Q}_{--}(\nu,t) + \int_{z=0}^{1} \hat{Q}_{-+}(\nu,z)\hat{\Psi}(z,t)du.\end{aligned}$$

$\hat{\Psi}(s,t)_{ij}$ – Conditioning on highest peak y

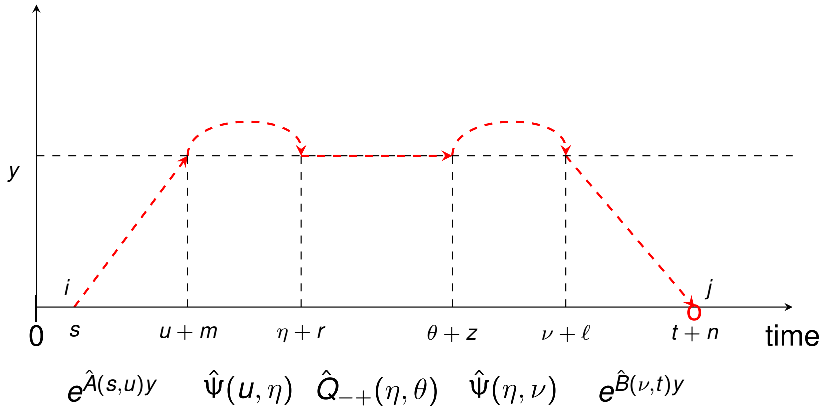
Here, $m, \ell, n = 0, 1, 2 \ldots$.



$e^{\hat{A}(s,u)y}$ $\hat{Q}_{+-}(u,\nu)$ $e^{\hat{B}(\nu,t)y}$

$\hat{\Psi}(s,t)_{ij}$ – Conditioning on lowest trough y

Here, $m, r, z, \ell, n = 0, 1, 2 \ldots$.



Matrix $\hat{U}_1(s,t)$

For all s, t with $0 \leq s \leq t$, define $U_1(s,t)$ which solves

$$\begin{aligned}
\frac{\partial}{\partial t} U_1(s,t) &= U_1(s,t) \begin{bmatrix} T_{--}(t) & T_{-0}(t) \\ T_{0-}(t) & T_{00}(t) \end{bmatrix} \\
\frac{\partial}{\partial s} U_1(s,t) &= -\begin{bmatrix} T_{--}(s) & T_{-0}(s) \\ T_{0-}(s) & T_{00}(s) \end{bmatrix} U_1(s,t) \\
U_1(s,s) &= \mathbf{I}
\end{aligned}$$

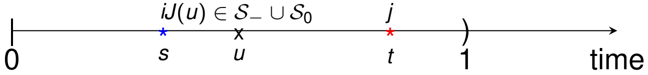
and let

$$\hat{U}_1(s,t) = \sum_{n=0}^{\infty} U_1(s,t+n) = (\mathbf{I} - U_1(s,s+1))^{-1} U_1(s,t).$$

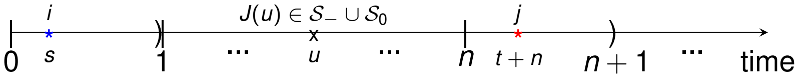
Physical interpretation of $\hat{U}_1(s,t)$

For $i, j \in \mathcal{S}_- \cup \mathcal{S}_0$,

$$[U_1(s,t)]_{ij} = P[J(t) = j \mid J(s) = i, J(u) \in \mathcal{S}_- \cup \mathcal{S}_0, s \le u \le t]$$



$$[\hat{U}_1(s,t)]_{ij} = \sum_{n=0}^{\infty} [U_1(s, t+n)]_{ij}$$

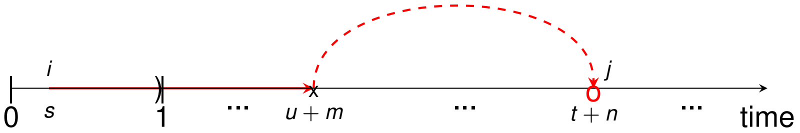


A related discrete-time Markov chain (DTMC)

Consider a DTMC on $[0,1) \times \mathcal{S}_-$ observed at the epochs when the CSFM hits level 0 from above, with $P = [P(s,t)_{ij}]_{s,t\in[0,1),i,j\in\mathcal{S}}$ given by

$$P(s,t)dt = \left[\begin{array}{cc} \mathbf{I} & \mathbf{0} \end{array}\right] \int_0^1 \hat{U}_1(s,u) \left[\begin{array}{c} T_{-+}(u) \\ T_{0+}(u) \end{array}\right] \hat{\Psi}(u,t)du$$

Here, $m, n = 0, 1, 2 \ldots$.



$\hat{U}_1(s,u)$ $\left[\begin{array}{c} T_{-+}(u) \\ T_{0+}(u) \end{array}\right]$ $\hat{\Psi}(u,t)$

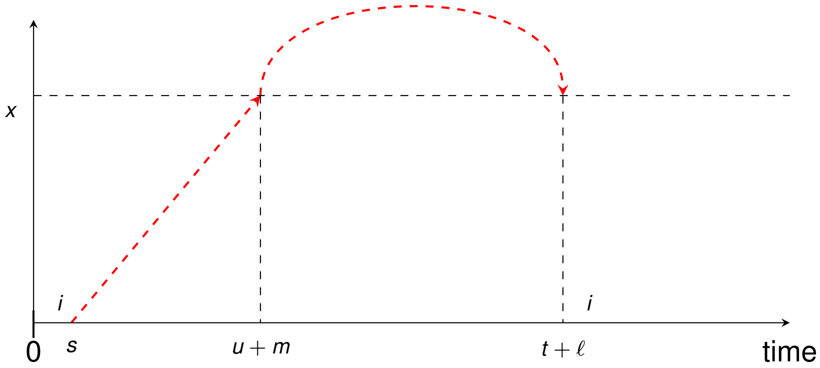
Asymptotic periodic distribution $\xi(t)$ of such DTMC

Let $\xi(t)$, $t \in [0, 1)$, be the unique solution of the set of equations

$$\begin{aligned} \int_{s=0}^{1} \xi(s) P(s,t)_{ij} ds &= \xi(t), \\ \int_{t=0}^{1} \xi(t) dt \mathbf{1} &= 1. \end{aligned}$$

Conditioning used in the derivation of $\pi(t;x)_-$

Here, $m, \ell = 0, 1, 2, \ldots$.



$$\begin{bmatrix} p_-(s) & p_0(s) \end{bmatrix} \begin{bmatrix} T_{-+}(s) \\ T_{0+}(s) \end{bmatrix} e^{\hat{A}(s,u)x} \qquad \hat{\Psi}(u,t)du(\mathbf{C}_-)^{-1}$$

Theorem

For all $t \in [0,1)$ we have

$$\left[\begin{array}{cc} p_-(t) & p_0(t) \end{array}\right] = \alpha \int_{s=0}^{1} \left[\begin{array}{cc} \xi(s) & \mathbf{0} \end{array}\right] \hat{U}_1(s,t)ds, \quad (1)$$

and

$$\left[\begin{array}{cc} \pi(t;x)_+ & \pi(t;x)_- \end{array}\right] = \int_{s=0}^{1} \left[\begin{array}{cc} p_-(s) & p_0(s) \end{array}\right] \left[\begin{array}{c} T_{-+}(s) \\ T_{0+}(s) \end{array}\right] \times \left[\begin{array}{cc} e^{\hat{A}(s,t)x}(\mathbf{C}_+)^{-1} & \int_{u=0}^{1} e^{\hat{A}(s,u)x}\hat{\Psi}(u,t)du(\mathbf{C}_-)^{-1} \end{array}\right] ds \quad (2)$$

and

$$\pi(t;x)_0 = \int_{s=0}^{1} \left[\begin{array}{cc} \pi(s;x)_+ & \pi(s;x)_- \end{array}\right] \left[\begin{array}{c} T_{+0}(s) \\ T_{-0}(s) \end{array}\right] \hat{U}_0(s,t)ds, \quad (3)$$

where $\alpha > 0$ is a normalizing constant that for all $t \in [0,1)$ solves

$$p(t)\mathbf{1} + \int_{x=0}^{\infty} \pi(t;x)dx\mathbf{1} = 1. \quad (4)$$

Normalising constant α

Let $(-A^{-1}(s,t)) = \int_{y=0}^{\infty} e^{A(s,t)y}dy$. Then

$$\alpha = \left(\int_{s=0}^{1} \begin{bmatrix} \xi(s) & \mathbf{0} \end{bmatrix} \hat{U}_1(s,t)ds\mathbf{1} \right.$$
$$+ \int_{s=0}^{1}\int_{u=0}^{1} \begin{bmatrix} \xi(u) & \mathbf{0} \end{bmatrix} \hat{U}_1(u,s) \begin{bmatrix} T_{-+}(s) \\ T_{0+}(s) \end{bmatrix} du$$
$$\times \begin{bmatrix} A^{-1}(s,t)(\mathbf{C}_+)^{-1} & \int_{v=0}^{1} A^{-1}(s,v)\hat{\Psi}(v,t)dv(\mathbf{C}_-)^{-1} \end{bmatrix} ds\mathbf{1}$$
$$+ \int_{s=0}^{1}\int_{u=0}^{1}\int_{v=0}^{1} \begin{bmatrix} \xi(v) & \mathbf{0} \end{bmatrix} \hat{U}_1(v,u) \begin{bmatrix} T_{-+}(u) \\ T_{0+}(u) \end{bmatrix} dv$$
$$\times \begin{bmatrix} -A^{-1}(u,s)(\mathbf{C}_+)^{-1} & -\int_{w=0}^{1} A^{-1}(u,w)\hat{\Psi}(w,s)dw(\mathbf{C}_-)^{-1} \end{bmatrix} du$$
$$\left. \times \begin{bmatrix} T_{+0}(s) \\ T_{-0}(s) \end{bmatrix} \hat{U}_0(s,t)ds\mathbf{1} \right)^{-1}.$$

Current work

- Proofs, numerical schemes and examples.
- Other cyclic SFMs of interest:
  - Cycle duration constant or with some distribution
  - SFMs with a moving lower boundary, which drops to 0 at the end of a cycle
  - SFMs which restarts at level $B > 0$ at the start of a cycle, if the level falls below some threshold level $0 < b < B$

References

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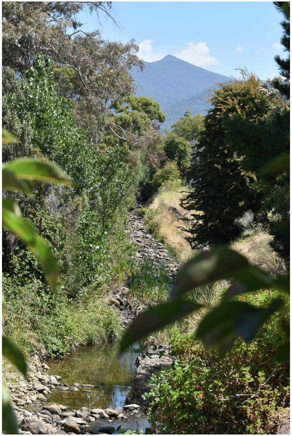
Thank you



We found Ψ!

Peter found Ψ!